AGN COVARIANCE FUNCTION DERIVATIONS

CONTENTS

INTRODUCTION

Though the core of the SPEAR methodology is couched in correlated statistics and linear algebra, it still invokes a physical understanding of the system in the stochastic DRW process, and the transfer functions by which this process affects the response curves. These manifest in the elements of the signal covariance matrix, S.

Our entire understanding of the physics at play in an AGN is encoded within this matrix, and so defining its elements is a task of great significance. Unfortunately, through the calculation of these elements is conceptually simple, it has a habit of becoming extremely involved upon contact with reality.

In this document, we derive and summarize the line-line, line-self and line-continuum covariances for any arbitrary pair of measurement times: gearing these results towards the reader being able to both understand and apply them with the least possible inconvenience.

THE LINE-LINE CORRELATION FROM CONVOLUTION

Before we get into the specifics of the tophat transfer function and its results, we'll take a brief aside to examine an alternative approach: making use of convolutions to describe the covariances in a general sense. The piecewise nature of the DRW auto-correlation function makes this approach less than useful in practice, and so we dedicate only this short segment to introduce the concept.

The most general description of the covariance between two measurements on two lines is:

$$C_{ab}(t_i,t_j) = \langle s_a(t_i)s_b(t_j)\rangle = C_{ab}(t_i,t_j) = \iint \langle s_c(t')s_c(t'')\rangle \psi_a(t_i-t')\psi_b(t_j-t'') \cdot dt'dt''$$

Notice that, if we define $\psi^A(t) = \psi(-t)$, this looks more like:

$$C_{ab}(t_i,t_j) = \iint \langle s_c(t')s_c(t'')\rangle\, \psi_a^A(t'-t_i)\psi_b^A(t''-t_j) \cdot dt'dt''$$

Which is something akin to the integral of a convolution:

> *The covariance between the correlation function and the transfer functions as we shift the transfer functions left and right.*

For brevity, we may write $\Phi(t',t'') = \langle s_c(t')s_c(t'')\rangle$, so that this appears as:

$$C_{ab}(t_i,t_j) = \int \left(\int \Phi(t',t'') \cdot \psi_a^A(t'-t_i)\, dt' \right) \cdot \psi_b^A(t''-t_j)\, dt''$$

Because of the integration boundaries being infinite, the inside integral depends only on:

$$t^* = t'' - t_i$$

Such that we may write this as:

$$C_{ab}(t_i,t_j) = \int f(t''-t_i) \cdot \psi_b^A(t''-t_j)\, dt''$$

Performing a variable change, and defining $\Delta t = t_j - t_i$:

$$C_{ab}(t_i,t_j) = \int f(t^*) \cdot \psi_b^A(t^*-\Delta t)\, dt^*$$

In fact, the final value, $\phi(t_i t_j)$, should **also** only depend on the difference in the times of interest. Thus, a valid general description would be to say:

$$C_{ab}(\Delta t) = \int f(t^*) \cdot \psi_b^A(t^*-\Delta t)\, dt^*$$

In this form, we can see the entire double integral as a sort of convolution. This may have the potential to provide much more elegant solutions to the covariance functions in cases beyond the scope of this report, but unfortunately introduces more problems than it solves in the heavily segmented cases we deal with in DRW's.

SOLUTIONS FOR TOPHAT FUNCTIONS

The piecewise nature of the DRW covariance function and the tophat transfer functions means that the convolution approach becomes extremely unwieldy. An alternative approach is to use the fact that the tophat function manifest as integration boundaries to create a simpler approach for that specific case.

A CRUDE APPROACH TO THE GENERAL CASE

Remember firstly that the covariance between the continuum curve, $s_c(t)$, and itself, is:

$$\langle s_c(t_i) s_c(t_j) \rangle = \sigma_\infty^2 \exp\left(-\frac{|t_i - t_j|}{\tau}\right)$$

Where τ is the characteristic damping timescale of the continuum DRW. Thus, the correlation is:

$$C_{ab}(t_i, t_j) = \sigma_\infty^2 \iint e^{-\frac{|t_a - t_b|}{\tau}} \psi_a(t_i - t_a) \psi_b(t_j - t_b) \cdot dt_a dt_b$$

Now, notice that the tophat transfer functions serve only to apply integration boundaries. If we define:

$$t_{a1} = t_i - [\tau_{0a} + w_a] \qquad t_{b1} = t_j - [\tau_{0b} + w_b]$$
$$t_{a2} = t_i - [\tau_{0a}] \qquad t_{b2} = t_j - [\tau_{0b}]$$

If the tophats have heights h_a and h_b, we get:

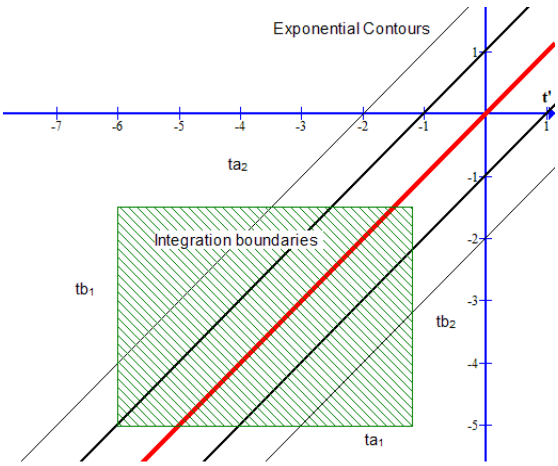
$$C_{ab} = \sigma_\infty^2 h_a h_b \int_{t_{a1}}^{t_{a2}} \int_{t_{b1}}^{t_{b2}} e^{-\frac{|t_a - t_b|}{\tau}} \cdot dt_a dt_b$$

This is still somewhat complicated by the piecewise nature of the exponential, but we can make things easier by dividing it into three possible sections:

1. Both integration boundaries above the $t_a = t_b$ line
2. One integration boundary above $t_a = t_b$ the other below
3. Both integration boundaries below $t_a = t_b$

Such that the total integral is just the sum of each:

$$C_{ab} \propto I_1 + I_2 + I_3$$



Notice that, as an integral of a strictly positive function over a positive domain, each region's ub integral, and thus the covariance as a whole is **always positive**.

The boundaries on each the regions of these are:

Region 1	**Region 2**	**Region 3**
$L_1 = t_{a1}$	$L_2 = \max(t_{b1}, t_{a1})$	$L_3 = \max(t_{b2}, t_{a1})$
$R_1 = \min(t_{a2}, t_{b1})$	$R_2 = \min(t_{a2}, t_{b2})$	$R_3 = t_{a2}$

If a region's right hand side occurs before its left-hand side, this indicates the region doesn't exist, and may be discarded:

$$if\ R_i < L_i, \qquad I_i = 0$$

Otherwise:

$$I_i = \int_{L_i}^{R_i} f_i(t_a)\, dt_a, \qquad f_i(t_a) = \int_{t_{b1}}^{t_{b2}} e^{-\frac{|t_a - t_b|}{\tau}}\, dt_b$$

The function $f_i(t_a)$ is the cumulative distribution function of a "Laplace Distribution", which has an established solution:

$$\int_{x_1}^{x_2} e^{-\frac{|x-\mu|}{b}} = b \begin{cases} e^{\frac{x-\mu}{b}} & x < \mu \\ 2 - e^{-\frac{x-\mu}{b}} & x \geq \mu \end{cases}$$

By dividing the integral into the three regions, we sidestep the bulk of the trouble caused by the piecewise nature of the CDF.

Region 1

$$f_1(t_a) = \tau\left[\left(2 - e^{-\frac{t_{b2}-t_a}{\tau}}\right) - \left(2 - e^{-\frac{t_{b1}-t_a}{\tau}}\right)\right]$$
$$= -\tau\left[e^{\frac{-t_{b2}}{\tau}} - e^{\frac{-t_{b1}}{\tau}}\right] e^{\frac{t_a}{\tau}}$$

Region 2

$$f_2(t_a) = \tau\left[\left(2 - e^{-\frac{t_{b2}-t_a}{\tau}}\right) - e^{\frac{t_{b1}-t_a}{\tau}}\right]$$
$$= \tau\left[2 - e^{-\frac{t_{b2}}{\tau}} e^{\frac{t_a}{\tau}} - e^{\frac{t_{b1}}{\tau}} e^{\frac{-t_a}{\tau}}\right]$$

Region 3

$$f_3(t_a) = \tau\left[e^{\frac{t_{b2}-t_a}{\tau}} - e^{\frac{t_{b1}-t_a}{\tau}}\right]$$
$$= \tau\left[e^{\frac{t_{b2}}{\tau}} - e^{\frac{t_{b1}}{\tau}}\right] e^{-\frac{t_a}{\tau}}$$

From here, finding each integral is just a matter of integrating each function over t_a between the given boundaries, and adding together:

$$I_1 = -\tau^2\left[e^{\frac{-t_{b2}}{\tau}} - e^{\frac{-t_{b1}}{\tau}}\right]\left(e^{\frac{R_1}{\tau}} - e^{\frac{L_1}{\tau}}\right)$$
$$I_2 = \tau^2\left[2\frac{(R_2 - L_2)}{\tau} - e^{-\frac{t_{b2}}{\tau}}\left(e^{\frac{R_2}{\tau}} - e^{\frac{L_2}{\tau}}\right) + e^{\frac{t_{b1}}{\tau}}\left(e^{\frac{-R_2}{\tau}} - e^{\frac{-L_2}{\tau}}\right)\right]$$
$$I_3 = -\tau^2\left[e^{\frac{t_{b2}}{\tau}} - e^{\frac{t_{b1}}{\tau}}\right]\left(e^{\frac{-R_3}{\tau}} - e^{\frac{-L_3}{\tau}}\right)$$

This is a entirely complete description of how to get the covariances, but is a little unwieldy to apply. In the following sections, we'll get simpler forms for specific cases by using a few tips and tricks.

TOWARDS A MORE ANALYTICAL SOLUTION

Though the methodical approach we've described is functional and robust, it doesn't provide much in the way of understanding. By making use of some of the system's symmetries, we can arrive at more concise results for specific cases. We have a couple of tools at our disposal:

We can shift the integration boundaries

The symmetry of the system means that we're free to translate the transfer "box" parallel to the $t_a = t_b$ line without changing the results. This means we're free to add or subtract any constant to all the integration boundaries at once. This lets us fix one of them equal to zero, for example t_{b1}:

$$t_{a1} = (t_i - t_j) - (\tau_{0a} - \tau_{0b}) - (w_a - w_b) \qquad t_{b1} = 0$$
$$t_{a2} = (t_i - t_j) - (\tau_{0a} - \tau_{0b}) - w_b \qquad t_{b2} = w_b$$

We can swap the transfer functions

Covariance is symmetrical, $C_{ab}(t_i, t_j) = C_{ab}(t_i, t_j)$, meaning we're free to enforce any *one* condition on the transfer functions and measurements times, e.g.

$$t_i \geq t_j, \qquad w_b < t_a$$

If this isn't the case, we can simply swap the values around without changing the answer we get.

We'll go through different special cases in order of increasing complexity, to help ease the reader into the working. If you're looking *only* to apply the results, jump straight to the "summary" or "General case" equations.

CONTINUUM-LINE TRANSFER FUNCTION

The continuum-line correlation can be recovered by setting $\psi_b(t) = \delta(t)$, i.e. $\tau_{0b} = 0$ and taking the limit of $w_b \to 0$. Doing so gives us the working variables:

$$t_a = t_i \qquad \Bigg| \qquad \begin{aligned} t_{b1} &= t_j - [\tau_{0b} + w_b] \\ t_{b2} &= t_j - [\tau_{0b}] \end{aligned}$$

And provides us with a relatively simple answer:

$$\langle s_l(t_i) s_b(t_j) \rangle = \sigma_\infty^2 h\tau \begin{cases} \left[e^{\frac{-t_{b1}}{\tau}} - e^{\frac{-t_{b2}}{\tau}} \right] e^{\frac{t_a}{\tau}} & t_a < t_{b1} \\ 2 - e^{-\frac{t_{b2}}{\tau}} e^{\frac{t_a}{\tau}} - e^{\frac{t_{b1}}{\tau}} e^{\frac{-t_a}{\tau}} & t_{b1} < t_a < t_{b2} \\ \left[e^{\frac{t_{b2}}{\tau}} - e^{\frac{t_{b1}}{\tau}} \right] e^{-\frac{t_a}{\tau}} & t_a > t_{b2} \end{cases}$$

As with the line-line case, we can make use the symmetry of the system, this time to enforce $t_a = 0$. Doing so give us:

$$\langle s_l(t_i) s_b(t_j) \rangle = \sigma_\infty^2 h\tau \begin{cases} e^{\frac{-t_{b1}}{\tau}} - e^{\frac{-t_{b2}}{\tau}} & 0 < t_{b1} \\ 2 - e^{-\frac{t_{b2}}{\tau}} - e^{\frac{t_{b1}}{\tau}} & t_{b1} < 0 < t_{b2} \\ e^{\frac{t_{b2}}{\tau}} - e^{\frac{t_{b1}}{\tau}} & 0 > t_{b2} \end{cases}$$

Where:

$$t_{b1} = (t_j - t_i) - [\tau_{0b} + w_b] \qquad\qquad t_{b2} = (t_j - t_i) - [\tau_{0b}]$$

For the sake of simplicity, let's define:

$$\Delta u = -(t_i - t_j) - \left[\tau_{0b} + \frac{w_b}{2}\right] \geq 0$$

Such that the boundaries are written:

$$t_{b1} = \Delta u - \frac{w_b}{2} \qquad\qquad t_{b2} = \Delta u + \frac{w_b}{2}$$

And the results look like:

$$\langle s_l(t_i) s_b(t_j) \rangle = 2\sigma_\infty^2 h\tau \begin{cases} 1 - e^{\frac{-w}{2\tau}} \cosh\left(\frac{\Delta u}{\tau}\right) & |\Delta u| < \frac{w}{2} \\ e^{\frac{-\Delta u}{\tau}} \sinh\left(\frac{w}{2}\right) & |\Delta u| > \frac{w}{2} \end{cases}$$

As before, we can see that this depends on the **difference** in times only, as expected.

LINE-LINE SELF TRANSFER FUNCTION

A particularly useful case is that of a line response's covariance with itself. This can be found by setting $\Delta u = \Delta t$ and $\Delta w = 0$, but we'll re-derive the result here from first principles as a matter of validation.

$$\psi_a(t) = \psi_b(t)$$

In this case, the (unshifted) working variables look like:

$$t_{a1} = t_i - [\tau_0 + w] \qquad\qquad t_{b1} = t_j - [\tau_0 + w]$$
$$t_{a2} = t_i - [\tau_0] \qquad\qquad t_{b2} = t_j - [\tau_0]$$

We begin by enforcing $t_{a1} = 0$ to get the much simpler integration boundaries:

$$t_{a1} = 0 \qquad\qquad t_{b1} = t_j - t_i$$
$$t_{a2} = w \qquad\qquad t_{b2} = t_j - t_i + w$$

We're free to switch the two measurements to enforce $\Delta t \geq 0$, thereby ensuring that all of the variables are positive. We then have only two possible cases:

Case 1: Small Times $\Delta t < w$ Region 2 only

Case 2: Moderate Times $\Delta t > \mathrm{w}$ Regions 1 and 2

We'll deal with each of these independently. Our results should be expected to converge to each other at $|\Delta t| = 0$.

We label the first case as being a "moderate" time difference in keeping with the naming in the general cases.

CASE 1: MODERATE TIME DIFFERENCES

In this case, we have the conditions:

$$t_{a1} < t_{b1} < t_{a2} < t_{b2}, \qquad |\Delta t| < w$$

Such that the region boundaries become:

Region 1	**Region 2**	**Region 3**
$L_1 = 0$	$L_2 = t_{b1}$	$L_3 = t_{b2}$
$R_1 = t_{b1}$	$R_2 = t_{a2}$	$R_3 = t_{a2}$

Notice that $t_{b2} > t_{a2}$, so region 3 doesn't exist. This leaves us only with regions 1 and 2. We're then free to plug in our integral boundaries to get:

$$I_1 = -\tau^2 \left[e^{\frac{-|\Delta t|-w}{\tau}} - e^{\frac{-|\Delta t|}{\tau}}\right]\left(e^{\frac{|\Delta t|}{\tau}} - 1\right)$$
$$I_2 = \tau^2 \left[2\frac{(w - |\Delta t|)}{\tau} - e^{-\frac{|\Delta t|+w}{\tau}}\left(e^{\frac{w}{\tau}} - e^{\frac{|\Delta t|}{\tau}}\right) + e^{\frac{|\Delta t|}{\tau}}\left(e^{\frac{-w}{\tau}} - e^{\frac{-|\Delta t|}{\tau}}\right)\right]$$

The interim working is involved and algebraically inconvenient, but eventually results in a much friendlier equation:

$$I = \frac{(w - |\Delta t|)}{\tau} + e^{\frac{-w}{\tau}} \cosh\left(\frac{|\Delta t|}{\tau}\right) - e^{\frac{-\Delta u}{\tau}}$$

CASE 2: LARGER TIME DIFFERENCES

In this case, we have the conditions:

$$t_{a1} < t_{a2} < t_{b1} < t_{b2}, \qquad |\Delta t| > w$$

Such that the region boundaries become:

Region 1	**Region 2**	**Region 3**
$L_1 = 0$	$L_2 = t_{b1}$	$L_3 = t_{b2}$
$R_1 = t_{a2}$	$R_2 = t_{a2}$	$R_3 = t_{a2}$

Notice that $t_{b2} > t_{b1} > t_{a2}$, so regions 2 and 3 don't exist. This leaves us only with region 1, which has well defined boundaries:

$$I = -\tau^2 \left[e^{\frac{-\Delta t - w}{\tau}} - e^{\frac{-\Delta t}{\tau}}\right]\left(e^{\frac{w}{\tau}} - 1\right)$$

As before, we're free to do some rearranging to get a simpler expression:

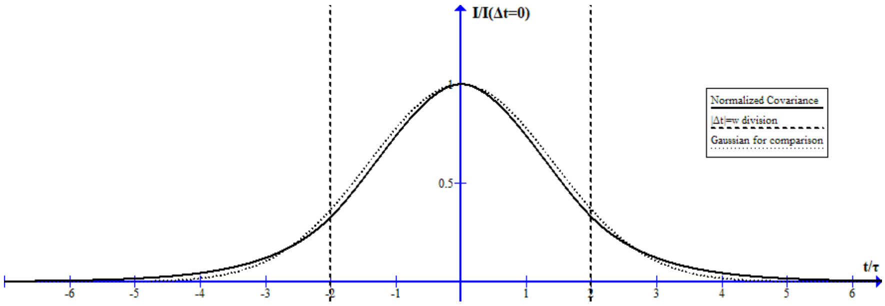
$$I = 2\tau^2 e^{\frac{-\Delta t}{\tau}}\left(\cosh\left(\frac{w}{\tau}\right) - 1\right)$$

SUMMARIZING, VALIDATING AND APPLYING

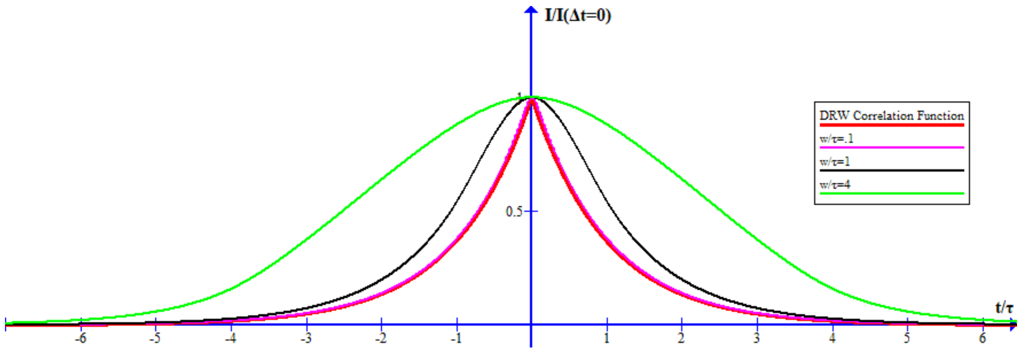
To summarize, the integral 'I' is, in total:

$$I = 2\tau^2 \begin{cases} \frac{(w-|\Delta t|)}{\tau} + e^{\frac{-w}{\tau}} \cosh\left(\frac{|\Delta t|}{\tau}\right) - e^{\frac{-\Delta u}{\tau}} & |\Delta t| < w \\ e^{\frac{-|\Delta t|}{\tau}} \left(\cosh\left(\frac{w}{\tau}\right) - 1\right) & |\Delta t| > w \end{cases}$$

Graphing this (and normalizing), we can see that it fits smooth, continuous and strictly positive bell-curve like shape that very **almost** fits a Gaussian;



We can also validate this result by seeing how to converges to the limit of the DRW's inherent correlation function in the limit $w \to 0$, i.e. $\psi(t) \to \delta(t-\tau)$:



Above, we've normalized the integrals such that they represent the line-self autocorrelation functions rather than their autocovariances. In this way, we can see that:

$$I_{a0} = I_a(|dt| = 0) = \sigma_a^2$$

Remembering that an alternative definition for this variance is:

$$C_a(\Delta t = 0) = \sigma_a^2 = \sigma_\infty^2 h_a^2 I_{a0}$$

We also arrive at an expression for tophat function height $h_a$that we can substitute into the line-line and line-continuum covariances:

$$h_a = \frac{\sigma_a}{\sigma_\infty}\sqrt{\frac{1}{I_{a0}}}, \qquad I_{0a} = 2\tau^2\left[\frac{w}{\tau} + e^{-\frac{w}{\tau}} - 1\right]$$

This allows us to replace h_a with as σ_aa model parameter, making for a much easier estimate.

SIMPLIFYING THE GENERAL COVARIANCE FUNCTION

We can make use of our "tips and tricks" to change our integration boundaries to be:

$t_{a1} = 0$	$t_{b1} = \Delta u - \Delta w$
$t_{a2} = \overline{w} - \Delta w$	$t_{b2} = \Delta u + \overline{w}$

Where we've defined:

$$\Delta u = \left| -[t_i - t_j] - [\tau_b - \tau_a] - \frac{[w_b - w_a]}{2} \right|, \qquad \overline{w} = \frac{w_b + w_a}{2}, \qquad \Delta w = \left| \frac{w_b - w_a}{2} \right|$$

We're free to enforce Δu, the distance between the "midpoint" of the two tophats, to be greater than zero. Though we technically aren't "allowed" to fix $\Delta w \geq 0$ by taking its absolute value, the result ands up being symmetrical about $\Delta w = 0$, and so we can do this for convenience without it affecting our final result.

In this way, we know that there are only three possible cases:

Case 1: Small Times	$\Delta u < \Delta w$	Region 2 only
Case 2: Moderate Times	$\Delta w < \Delta u < \overline{w}$	Regions 1 and 2
Case 3: Large Times	$\overline{w} < \Delta u$	Region 1 only

We'll deal with each of these in sequence to see their results.

CASE 1: SMALL TIMES

Region 1

$$L_1 = 0$$

$$R_1 = \overline{w} - \Delta w$$

This integral is region 1 only, and the boundaries are already readily defined. As such, the result comes out almost directly:

$$I = \tau^2 e^{-\frac{\Delta u}{\tau}} \left[e^{\frac{-\overline{w} + \Delta w}{\tau}} - 1 \right] \left(e^{\frac{\overline{w} + \Delta w}{\tau}} - 1 \right)$$

This may also be written in a more intuitive form:

$$I = 2\tau^2 e^{\frac{-\Delta u}{\tau}} \left[\frac{\overline{w} - |\Delta w|}{\tau} + \left(e^{\frac{-\overline{w}}{\tau}} - e^{\frac{-|\Delta w|}{\tau}} \right) \cosh\left(\frac{\Delta u}{\tau} \right) \right]$$

CASE 2: MODERATE TIMES

Region 1

$$L_1 = 0$$

$$R_1 = \Delta u - \Delta w$$

Region 2

$$L_2 = \Delta u - \Delta w$$

$$R_2 = \overline{\mathrm{w}} - \Delta w$$

In this case, we have regions 1 and 2 present, with the above boundaries:

$$I_1 = -\tau^2 \left[e^{\frac{-\Delta u - w_b}{\tau}} - e^{\frac{-\Delta u}{\tau}} \right] \left(e^{\frac{\Delta u}{\tau}} - 1 \right)$$

$$I_2 = \tau^2 \left[2 \frac{(w_a - \Delta u)}{\tau} - e^{\frac{-\Delta u - w_b}{\tau}} \left(e^{\frac{w_a}{\tau}} - e^{\frac{\Delta u}{\tau}} \right) + e^{\frac{\Delta u}{\tau}} \left(e^{\frac{-w_a}{\tau}} - e^{\frac{-\Delta u}{\tau}} \right) \right]$$

Adding these together and rearranging:

$$I = 2\tau^2 \left[\frac{\overline{\mathrm{w}} - \Delta u}{\tau} + e^{\frac{-\overline{w}}{\tau}} \cosh(\Delta \mathrm{u}) - e^{\frac{-\Delta u}{\tau}} \cosh\left(\frac{\Delta w}{\tau} \right) \right]$$

CASE 3: LARGE TIMES

Region 2

$$L_2 = 0$$

$$R_2 = \overline{w} - \Delta w$$

This has only one region, and so the result is relatively easy to calculate:

$$I_2 = \tau^2 \left[2 \frac{\overline{\mathrm{w}} - |\Delta w|}{\tau} - e^{-\frac{\Delta u + \overline{\mathrm{w}}}{\tau}} \left(e^{\frac{\overline{\mathrm{w}} - |\Delta w|}{\tau}} - 1 \right) + e^{\frac{\Delta u - |\Delta w|}{\tau}} \left(e^{\frac{-\overline{\mathrm{w}} + |\Delta w|}{\tau}} - 1 \right) \right]$$

After some expanding and rearranging, we get:

$$I = 2\tau^2 \left[e^{\frac{-\Delta u}{\tau}} [\cosh(\overline{\mathrm{w}}) - \cosh(\Delta w)] \right]$$

SUMMARY

Collecting all three cases together, we have a piecewise function:

$$I = 2\tau^2 \begin{cases} \dfrac{\overline{w} - |\Delta w|}{\tau} + \left(e^{\frac{-\overline{w}}{\tau}} - e^{\frac{-|\Delta w|}{\tau}}\right)\cosh\left(\dfrac{\Delta u}{\tau}\right) & \Delta u < |\Delta w| \\ \dfrac{\overline{w} - \Delta u}{\tau} + e^{\frac{-\overline{w}}{\tau}}\cosh\left(\dfrac{\Delta u}{\tau}\right) - e^{\frac{-\Delta u}{\tau}}\cosh\left(\dfrac{\Delta w}{\tau}\right) & |\Delta w| < |\Delta u| < \overline{w} \\ e^{\frac{-\Delta u}{\tau}}\left[\cosh\left(\dfrac{\overline{w}}{\tau}\right) - \cosh\left(\dfrac{\Delta w}{\tau}\right)\right] & \overline{w} < |\Delta u| \end{cases}$$

Where:

$$\Delta u = \left|-[t_i - t_j] - [\tau_b - \tau_a] - \frac{[w_b - w_a]}{2}\right|, \qquad \overline{w} = \frac{w_b + w_a}{2}, \qquad \Delta w = \left|\frac{w_b - w_a}{2}\right|$$

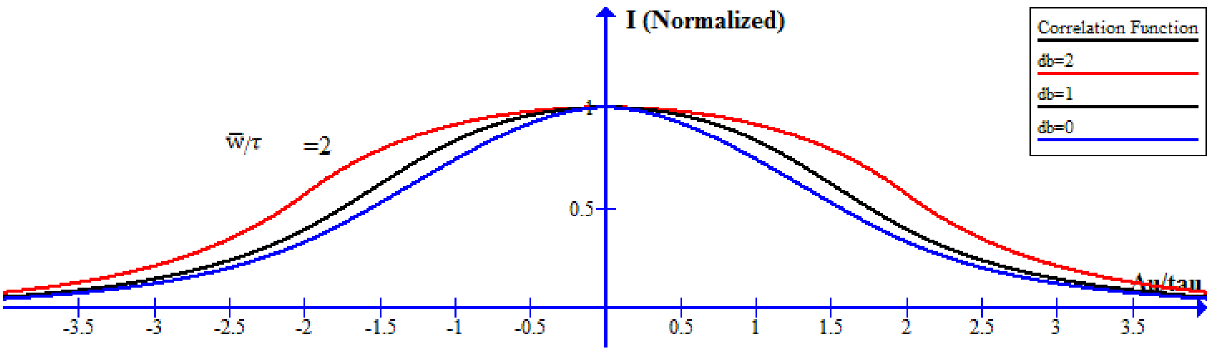
We can validate this by taking the case of $\psi_a = \psi_b$:

$$\Delta u = |\Delta t|, \qquad \Delta w = 0, \qquad \overline{w} = w$$

This makes the third branch impossible, as it would require $|\Delta t| < 0$. We then get:

$$I = 2\tau^2 \begin{cases} \dfrac{(w - |\Delta t|)}{\tau} + e^{\frac{-w}{\tau}}\cosh\left(\dfrac{|\Delta t|}{\tau}\right) - e^{\frac{-\Delta u}{\tau}} & |\Delta t| < w \\ e^{\frac{-|\Delta t|}{\tau}}\left(\cosh\left(\dfrac{w}{\tau}\right) - 1\right) & |\Delta t| > w \end{cases}$$

Which is exactly the result we found for self-self correlation.



Line-Line Covariance Integrals for Various Values of $|\Delta w|$

A COMPLETE SUMMARY FOR TOPHAT COVARIANCES

Applying everything we know, the general expression for covariance between any two function is:

$$\langle s_a(t_i), s_b(t_j)\rangle = \frac{\sigma_a\sigma_b}{\sqrt{I_{0a}I_{b0}}}\begin{cases} \dfrac{\overline{w}-|\Delta w|}{\tau} + \left(e^{\frac{-\overline{w}}{\tau}} - e^{\frac{-|\Delta w|}{\tau}}\right)\cosh\left(\dfrac{\Delta u}{\tau}\right) & \Delta u < |\Delta w| \\ \dfrac{\overline{w}-\Delta u}{\tau} + e^{\frac{-\overline{w}}{\tau}}\cosh\left(\dfrac{\Delta u}{\tau}\right) - e^{\frac{-\Delta u}{\tau}}\cosh\left(\dfrac{\Delta w}{\tau}\right) & |\Delta w| < |\Delta u| < \overline{w} \\ e^{\frac{-\Delta u}{\tau}}\left[\cosh\left(\dfrac{\overline{w}}{\tau}\right) - \cosh\left(\dfrac{\Delta w}{\tau}\right)\right] & \overline{w} < |\Delta u| \end{cases}$$

Where we have working variables:

$$\Delta u = \left|-[t_i - t_j] - [\tau_b - \tau_a] - \frac{[w_b - w_a]}{2}\right|, \qquad \Delta w = \frac{|w_b - w_a|}{2}, \qquad \overline{w} = \frac{[w_b + w_a]}{2}$$

And the normalizing coefficients:

$$I_{0a} = \left[\frac{w_a}{\tau} + e^{\frac{-w_a}{\tau}} - 1\right], \qquad I_{0b} = \left[\frac{w_b}{\tau} + e^{\frac{-w_b}{\tau}} - 1\right]$$

In the specific case of a lines self-correlation, we have the simpler expression:

$$\langle s_a(t_i), s_a(t_j)\rangle = \frac{\sigma_a^2}{I_{0a}}\begin{cases} \dfrac{(w-|\Delta t|)}{\tau} + e^{\frac{-w}{\tau}}\cosh\left(\dfrac{|\Delta t|}{\tau}\right) - e^{\frac{-\Delta u}{\tau}} & |\Delta t| < w \\ e^{\frac{-|\Delta t|}{\tau}}\left(\cosh\left(\dfrac{w}{\tau}\right) - 1\right) & |\Delta t| > w \end{cases}$$

Where:

$$\Delta t = |t_i - t_j|$$

And, in the case of a lines covariance with the continuum:

$$\langle s_l(t_i) s_a(t_j)\rangle = \frac{\sigma_\infty\sigma_a}{\sqrt{I_{0a}}}\begin{cases} 1 - e^{\frac{w}{2\tau}}\cosh\left(\dfrac{\Delta u}{\tau}\right) & |\Delta u| < \dfrac{w}{2} \\ e^{\frac{-\Delta u}{\tau}}\sinh\left(\dfrac{w}{2}\right) & |\Delta u| > \dfrac{w}{2} \end{cases}$$

Where:

$$\Delta u = \left|-[t_i - t_j] - \tau_c - \frac{w}{2}\right|$$

The resulting curves:

- Are symmetrical about $\Delta u = 0$
- Are symmetrical about $\Delta w = 0$
- Reach a maximum at $\Delta u = 0$
- Are broadened by increasing $\overline{w}$ and $|\Delta w|$