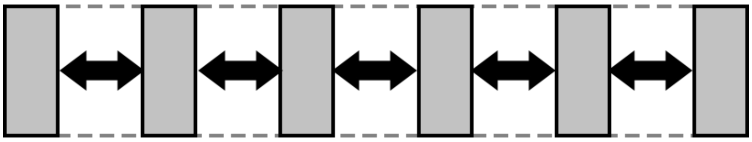
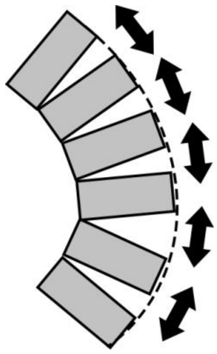
MAGNET ELASTICITY

The attractive force between magnets lined end-to-end means that they have a measure of "elasticity", keeping them "bound" together even when separated/extended under force:





This elasticity allows the row of magnets to support a stable curved shape (right).

In this document, we'll make an attempt to answer to questions:

1. How to we quantify this elasticity?
2. What is the smallest radius of curvature a row of magnets can support?

ELASTICITY OF A ROW OF MAGNETS

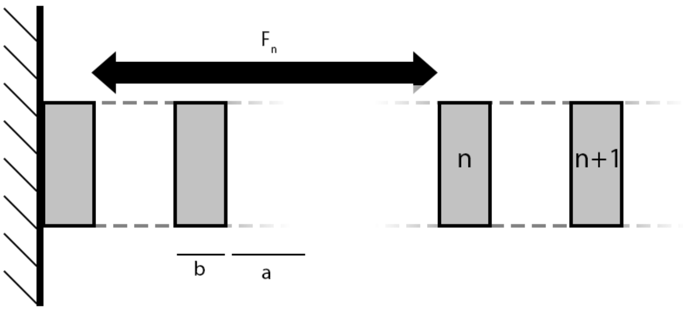
Let's start with the first question: what is the elasticity of a row of magnets? To describe this, we'll define elasticity as being "the relative increase in length under an applied tension":

$$E = \frac{1}{L}\frac{dL}{dT}$$

In the absence of electrical fields, we can ignore the complicated nature of magnets and treat them as simple non-induced dipoles, for which the axial dipole-dipole force scales with distance like:

$$F = \frac{k}{r^4}$$

Where 'r' is the distance between the two dipoles. In regards to geometry, let the magnets have widths b, and separation 'a', such that the distance between two dipoles 'n' magnets apart is $r_n = n[b + a]$:



In this way, the total tension force (i.e. the force in one particular direction) on a given magnet is:

$$F(a, N) = \frac{k}{[b + a]^4} \sum_{i=0}^{N} \frac{1}{n^4}$$

Now differentiating against 'a' to find how the force changes with magnet spacing:

$$\frac{dF}{da} = \frac{-4k}{[b + a]^5} \sum_{i=0}^{N} \frac{1}{n^4}$$

At rest, the magnets have zero separation, i.e. $a = 0$, which gives:

$$\frac{dF}{da} = -4\frac{k}{b^5}\sum_{i=0}^{N}\frac{1}{n^4}$$

Of particular note is the case of $N \to \infty$, in which the summation term goes to the limit $\frac{\pi^4}{90}$. This gives the total derivative as:

$$\frac{dF}{da} = -4\frac{k}{b^5}\frac{\pi^4}{90}$$

Now to use this to find elasticity. First remember that our definition is:

$$E = \frac{1}{L}\frac{dL}{dF}$$

We can use a change of variables by remembering that ‘a’ changes the total length L by:

$$dL = N \cdot da$$

Such that we can say:

$$E = \frac{N}{L}\left(\frac{dF}{da}\right)^{-1}$$

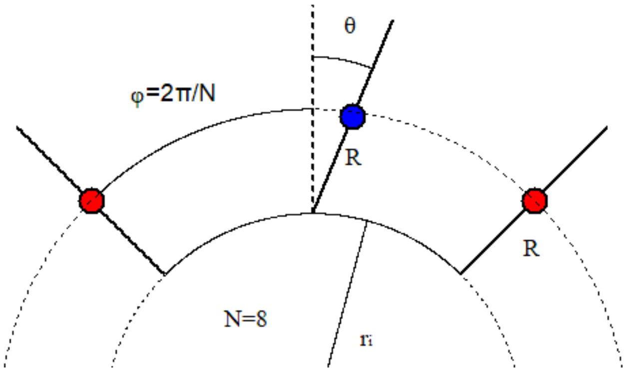
Leading to an overall elasticity expression of:

$$E = -\frac{45}{2\pi^5}\frac{b^6}{k}$$

A negative elasticity would be odd in normal materials mechanics, but makes sense in this context: the greater the separation between the magnets, the less they attract one another, and so the less restoring force they sustain.

MAGNET BENDING

The negative elasticity means that we can't use simple materials bending analogies to solve the question of the how tightly a row of magnets can bend. Instead, we'll look at a simplified model of the geometry to get some insight about the problem. Consider a set of 'N' magnets of half-width 'R', aligned corner-to-corner in a circle of inner radius r_i:



Simplified Model

At perfect alignment, the system symmetry means that the forces will all be balanced. As such, we'll instead define the system to be "stable" if it self-corrects a perturbation in the alignment of any one of the magnets, i.e. a small rotation θ produces a restoring moment instead of a diverging one:

$$\frac{dM_i}{d\theta_i} > 0$$

Our model relies on two main simplifications:

- The magnets can be treated as point dipoles at their geometric centers
- The dipole-dipole forces decay with distance fast enough that we can ignore all but the closest magnets

MOMENTS ACTING ON THE MAGNET

In general, for a magnet of moment 'm' subject to dipole magnetic field 'B', the forces and moments acting on it are:

$$\vec{F}_m = \nabla(\vec{m} \cdot \vec{B})$$

$$\vec{M}_m = (\vec{m} \times \vec{B})$$

If 'R' is the vector adjoining the dipole to its pivot point, this yields a total moment of:

$$\vec{M}_t = \vec{F} \times \vec{R} + \vec{M}_m$$

For a point-dipole, the generated magnetic field is:

$$B_i(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{3\vec{r}(\vec{m} \cdot \vec{r})}{|r|^5} - \frac{m}{|r|^3}\right)$$

Where r is the vector adjoining the point of interest and the magnets position.

Calculating Moments

Recall that the total moment acting on the magnet is:

$$\vec{M}_t = \nabla(\vec{m} \cdot \vec{B}) \times \vec{R} + (\vec{m} \times \vec{B})$$

We're after the **total** derivative of M_t against θ, i.e:

$$\frac{dM_t}{d\theta} = \frac{dx}{d\theta}\frac{\partial M_{tz}}{\partial x} + \frac{dy}{d\theta}\frac{\partial M_{tz}}{\partial y}$$

Notice that the positions that the pivoting magnet can take all follow:

$$\vec{r}_0 = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} R\sin(\theta) \\ r_i + R\cos(\theta) \end{pmatrix}$$

Also that the magnetic moment of the pivoting magnet is normal to this:

$$\vec{m}_0 = m \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix}$$

The other two magnets, meanwhile, have positions and moments:

$$\vec{r}_{ab} = (r_i + R) \begin{pmatrix} \sin(\pm\phi) \\ \cos(\phi) \end{pmatrix}, \qquad \vec{m}_{ab} = \begin{pmatrix} \cos(\phi) \\ \sin(\mp\phi) \end{pmatrix}$$

Over a sufficiently small angle change $d\theta$, we can say that the dipole's position changed by:

$$dy = 0, \qquad dx = Rd\theta$$

Such that the derivative of interest is:

$$\frac{dM_t}{d\theta} = R\frac{dM_t}{dx} = R\frac{d}{dx}\left(\nabla(\vec{m} \cdot \vec{B}) \times \vec{R} + (\vec{m} \times \vec{B})\right)$$

We'll these two terms apparent independently.

First Term: Moment Due To Magnet Torque

We know the moment vector of the perturbed magnet changes in direction, and we can resolve the applied magnetic field from its neighbours into x and y components:

$$\vec{m}_0 = m\begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \\ 0 \end{pmatrix}, \qquad \vec{B} = \begin{pmatrix} B_x \\ B_y \\ 0 \end{pmatrix}$$

Such that the magnitude of the applied moment is:

$$\frac{1}{m}|M_m| = \cos(\theta)\, B_y + \sin(\theta)\, B_x$$

Differentiating against θ:

$$\frac{1}{m}\frac{d}{d\theta}|M_m| = -\sin(\theta)\, B_y + \cos(\theta)\, B_x + R\cos(\theta)\frac{d}{dx}B_y + R\sin(\theta)\frac{d}{dx}B_x$$

Now applying at $\theta = 0$, and using the fact that $B_y = 0$ at this angle:

$$\left(\frac{d|M_m|}{d\theta}\right)_{\theta=0} = m\left[B_x + R\frac{d}{dx}B_y\right]$$

SECOND TERM: MOMENT DUE TO MAGNET FORCE

The second term, before differentiation, is:

$$M_f = \vec{R} \times \nabla(\vec{m} \cdot \vec{B})$$

In which the terms are (padded to 3 dimensions to allow us to take the cross product)

$$\vec{m}_0 = m \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \\ 0 \end{pmatrix}, \qquad \vec{B} = \begin{pmatrix} B_x \\ B_y \\ 0 \end{pmatrix}, \qquad R = \begin{pmatrix} R\sin(\theta) \\ R\cos(\theta) \end{pmatrix}$$

This leads to:

$$(\vec{m} \cdot \vec{B}) = m(\cos(\theta) B_x - \sin(\theta) B_y)$$

For gradient:

$$\nabla(\vec{m} \cdot \vec{B}) = \vec{m} \cdot \nabla(\vec{B}) = m \begin{pmatrix} \cos(\theta) \frac{d}{dx} B_x \\ -\sin(\theta) \frac{d}{dy} B_y \\ 0 \end{pmatrix}$$

Now taking the cross product's magnitude (third term)

$$\begin{aligned} -\frac{1}{m}|M_f| &= R\cos(\theta)\cos(\theta)\frac{d}{dx}B_x + R\sin(\theta)\sin(\theta)\frac{d}{dy}B_y \\ &= R\cos^2(\theta)\frac{d}{dx}B_x + R\sin^2(\theta)\frac{d}{dy}B_y \end{aligned}$$

And now differentiating against θ:

$$\frac{-1}{m}\frac{d}{d\theta}|M_f| = \left[(-2R\sin(2\theta))\frac{d}{dx}B_x + 2R\cos(2\theta)\frac{d}{dy}B_y\right] + \left[R(R\cos^2(\theta))\frac{d^2}{d^2x}B_x + R^2\sin^2(\theta)\frac{d^2}{dydx}B_y\right]$$

Applying at $\theta = 0$ and using $\frac{d}{dx}B_x = 0$ and $\frac{d}{dy}B_y = 0$ due to symmetry:

$$\frac{1}{m}\frac{d}{d\theta}|M_f| = -R^2\frac{d^2}{d^2x}B_x$$

RESULT

Combining the derivatives of our pure moment and force-moment, we get the much simpler expression:

$$\frac{dM}{d\theta} = m\left[B_x + R\frac{d}{dx}B_y - R^2\frac{d^2}{d^2x}B_x\right]$$

If we can find the three terms $(B_x, \frac{d}{dx}B_y\ and\ \frac{d^2}{d^2x}B_x)$ then we can easily find the sign of the perturbation response.

Recall that, from our geometry, our stability is defined:

$$\frac{dM}{d\theta} < 0, \qquad Unstable$$
$$\frac{dM}{d\theta} > 0, \qquad Stable$$

From the setup if the magnets, we have:

$$B_x > 0, \qquad \frac{dB_y}{dx} < 0, \qquad \frac{-d^2}{d^2x}B_x < 0$$

This matches with our intuition about the system:

- The first term describes the central magnet’s motivation to align parallel to the local magnet field
- The second term describes how this local magnetic field changes direction away from the axis
- The third term describes the direct axial attraction to the neighbouring magnets, and how this increases in strength as you get closer to them

So there are two stabilizing influences and one destabilizing. Finding the critical inner radii, we find that it is:

$$R = \frac{\frac{B_x}{R} + \frac{dB_y}{dx}}{\frac{d^2B_x}{d^2x}}$$

From this point forward, we’ll estimate the derivatives numerically rather than sticking with analytical solutions.

NUMERICAL SIMULATION

At this stage, we choose to solve the restoring moment problem numerically. Recall that our geometric setup has two magnets, 'L' and 'R', with:

$$m_L = m\begin{pmatrix}\cos(\phi)\\ \sin(\phi)\end{pmatrix} \;\; m_R = m\begin{pmatrix}\cos(\phi)\\ -\sin(\phi)\end{pmatrix}$$

$$r_L = (R + r_i)\begin{pmatrix}-\sin(\phi)\\ \cos(\phi)\end{pmatrix} \;\; r_R = (R + r_i)\begin{pmatrix}\sin(\phi)\\ \cos(\phi)\end{pmatrix}$$

Where $\phi = \frac{2\pi}{N}$, the angle separating the magnets at rest. Each magnet produces a magnetic dipole-field:

$$B_i(\vec{r}) = \frac{\mu_0}{4\pi}\left(\frac{3\vec{r}(\vec{m_i}\cdot\vec{r})}{|r|^5} - \frac{\vec{m}_i}{|r|^3}\right)$$

And we're interested in taking the field $\vec{B} = \vec{B}_L + \vec{B}_R$, and finding when the restoring moment goes to zero:

$$m\left[B_x + R\frac{d}{dx}B_y - R^2\frac{d^2}{d^2x}B_x\right] = 0$$

I.e. to get the roots of the function:

$$f(R, r_i, N) = B_x + R\frac{d}{dx}B_y - R^2\frac{d^2}{d^2x}B_x$$

The derivatives can be estimated by using finite difference methods:

$$\frac{d}{dx}B_y \approx \frac{B_y(\Delta x) - B_y(-\Delta x)}{2\Delta x}, \qquad \frac{d^2}{d^2x}B_x \approx \frac{B_x(\Delta x) - 2B_x(0) + B_x(-\Delta x)}{\Delta x^2}$$

Where we use a special step size proportional to the magnet width to avoid round-off error:

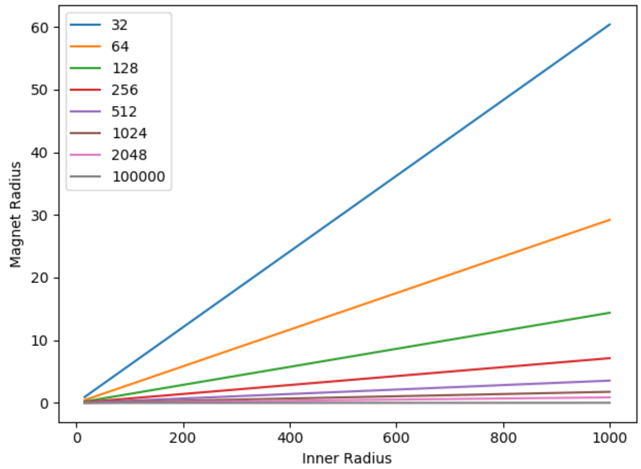
$$\Delta x \propto R$$

RESULTS

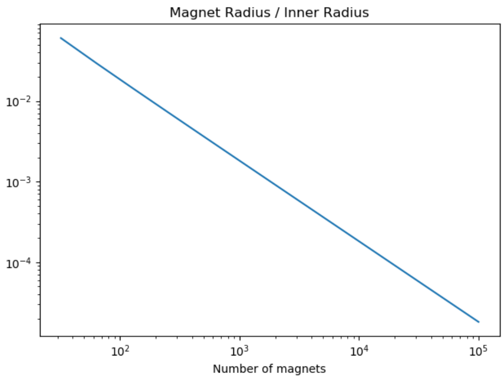
Solving this numerically for many different setups (see appendix) we find that the minimum stable inner radius is **directly proportional** to the magnet radius (right)

This makes sense: we know from dimensionless analysis that this stability should be a function of the **ratio** of inner radius and magnet radius, implying a direct proportionality, i.e.:

$$\frac{R}{r_i} = g(N)$$



Min. Circle Radius vs Magnet Half-Width



Unitless Circle Radius vs Magnet Number

As for how this maximum stable ratio changes with the number of magnets, we find that it is described extremely well by an exponential relationship:

$$\frac{R}{r_i} \leq A \cdot e^{kN}$$

Where:

$$A = 2.79 \cdot 10^{14}, \qquad k = -1.23$$

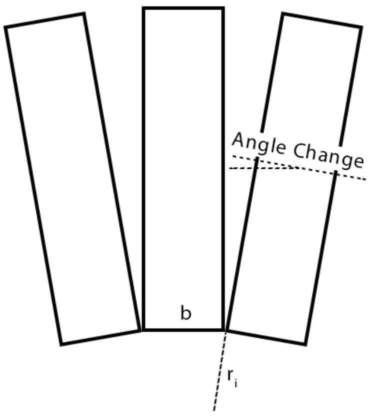
Note that this doesn't plateau to a constant value: the more magnets you have, the broader the circle has to be to support them.

This only provides a locus of possible stable configurations, we'd like to get things down to a single minimum value of r_i for a particular magnet geometry. We can do this by considering the "thickness" of the magnets, which we'll call 'b', to be related to the curvature of the circle as a whole:

$$\frac{1}{r_i} \approx \frac{\Delta\theta}{b} = \frac{2\pi}{Nb}$$

Combining this with our critical radius ratio equation, we get a more definite expression for the maximum circle curvature:

$$R \leq A r_i e^{2\pi k \frac{r_i}{b}}$$



Magnet Thickness & Curvature

APPENDIX: CODE

```
from math import pi, cos, sin
import numpy as np
from scipy.optimize import fsolve, bisect

'''
magsim02.py

Runs the numerical calculations for the
bending magnet problem.

    -Hugh McDougall, 2019
'''

def f(N,ri,R):
    '''
    Calculates the restoring moment derivative, dM/dtheta
    in arbitrary units for fixed set properties.

    Inputs
        N   int     Number of magnets in loop
        ri  float   Inner radius of circle
        R   float   Half-width/radius of magnets
    '''

    #Magnet separation angle
    thet=2*pi/N

    #Magnet position vectors
    r1=(R+ri)*np.array([-sin(thet),cos(thet)])
    r2=(R+ri)*np.array([0,1])
    r3=(R+ri)*np.array([sin(thet),cos(thet)])

    #Magnet separation vectors
    r12=r2-r1
    r32=r2-r3

    #Finite difference steps
    dx=.0001*R
    dy=dx

    #Magnet moment vectors
    m1=np.array([cos(thet), sin(thet)])
    m3=np.array([cos(thet),-sin(thet)])

    #Magnetic field function
    def B(r):
        rL=(r-r1)
        rR=(r-r3)

        BL=3*np.dot(rL,m1)/np.linalg.norm(rL)**5
        BL-=m1/np.linalg.norm(rL)**3

        BR=3*np.dot(rR,m3)/np.linalg.norm(rR)**5
        BR-=m3/np.linalg.norm(rR)**3

        return(BL+BR)

    #Calculate field and derivatives
    Bx=B(r2)[0]
```

```
    Byx=(B(r2+np.array([dx,0]))+B(r2+np.array([-dx,0])))[1]/dx
    Bxx=(B(r2+np.array([dx,0]))
         -2*B(r2)
         +B(r2+np.array([-dx,0])))[0]/dx**2

    return(Bx+R*Byx-R**2*Bxx)

#Test case
out=f(1000,1,.1)

#Plot results
import matplotlib.pylab as plt

#Finding/plotting stability contours
N   =[32,64,128,256,512,1024,2048,10**5]     #Magnet counts to test
rats=[]                                      #Stable ratios (output)
I   =64                                      #No. inner radii to test

Rin=np.linspace(0,1000,I+1)[1:]
Rout=Rin*0

for n in N:
    for i in range(I):
        rin=Rin[i]

        #Get bisection boundaries
        rout=rin
        while f(n,rin,rout)>0:
            rout/=2
        g=lambda rout: f(n,rin,rout)

        #Solve and output
        rout=bisect(g,rout,2*rout)
        Rout[i]=rout

    #Plot and output results
    plt.plot(Rin,Rout,label=str(n))
    print(n)
    rat=np.average(Rout/Rin)
    rats.append(rat)

print("Calcs done. Exctracting data")

#Plot R vs ri
plt.legend()
plt.xlabel("Inner Radius")
plt.ylabel("Magnet Radius")
plt.tight_layout()

#Plot ri/R vs N
fig = plt.figure()
ax = fig.add_subplot(1, 1, 1)
ax.set_yscale('log')
ax.set_xscale('log')
plt.plot(N,rats)
plt.xlabel("Number of magnets")
plt.title("Magnet Radius / Inner Radius")
plt.tight_layout()
plt.show()
```